CONSTRUCTION OF CONTINUOUS, INTEGRABLE FUNCTIONS WITH EXTREME BEHAVIOR AT INFINITY

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Abstract. For any real sequence $\{c_n\}$ with $c_n \to \infty$, this constructs a function f which is continuous and integrable on the real line, and such that for every real $x \neq 0$ $\limsup_{n\to\infty} c_n f(n\,x) = \infty$.

Let f be a continuous, integrable, real-valued function on the real numbers $\mathbb{R}$. Let $x \in \mathbb{R}$, $x \neq 0$, and consider what happens to $f(n\,x)$ as n tends to ∞ through the positive integers $\mathbb{N}^*$.[1] Emmanuel Lesigne [1] shows the following:

T1. Even if f is not continuous, $f(n\,x) \to 0$ for almost all x.

T2. For any sequence $\{c_n\}$ with $c_n \to \infty$, no matter how slowly, there is a non-negative, continuous, integrable function $f_{E'}$ with $\limsup_{n\to\infty} c_n f_{E'}(n\,x) = \infty$ for all x not in a set E_{K} of Lebesgue measure zero.

T3. In this, the condition $c_n \to \infty$ cannot be replaced by $\limsup_{n\to\infty} c_n = \infty$.

He asks whether the the second statement (T2) is true for *all* $x \neq 0$. In this paper we show that it is:

Theorem. *For any sequence* $\{c_n \,|\, n \in \mathbb{N}^*\}$ *with* $c_n \to \infty$ *there is a continuous, integrable function* f *such that* $\limsup_{n\to\infty} c_n f(n\,x) = \infty$ *for all* $x \neq 0$.

Our proof constructs a nonnegative, continuous, integrable function f_E which satisfies the theorem for all x in a set $E \supseteq E_{\mathrm{K}}$. Then $f := f_E + f_{E'}$ satisfies the theorem.[2]

In the proof T2, Lesigne uses Khinchin's Theorem (Theorem 32 of Khinchin's book [2]; see also, the Appendix, below). It is that theorem that provides the set E_{K}.

Khinchin's proof establishes that $E_{\mathrm{K}} \cap (0,1) \subset E^0_F \cup E^0_G$, where E^0_F and E^0_G (our notation) are sets constructed in the proofs of the book's Theorem 30 and 31, respectively, the superscript 0 indicating subsets of (0,1). Khinchin shows that E^0_F and E^0_G are sets of measure zero, and he notes that the result applies to all intervals of $\mathbb{R}$ through translation by integers; i.e., that $E_{\mathrm{K}} \subseteq \cup_{j=-\infty}^{\infty}(j + E^0_F \cup E^0_G)$.

The translation technique does not seem to apply here. If we knew that the theorem were true for $x \in (0,1)$, then extending it would require dealing with $f(n\,(x+j)) = f(n\,x + n\,j)$ where $j \in \mathbb{Z}$, the set of integers, and it is not clear how one can relate this to $f(n\,x)$. Therefore, we will take a different approach.

In the proofs of Theorems 30 and 31 of his book, Khinchin defines open subsets $E_{m,n}$ and $E_n(e^{A\,n})$ of $(0,1)$. This notation is not well suited to our purposes, so we will use F^0_{mn} and G^0_n for these sets, and we have $E^0_F := \cup_{m=1}^{\infty} \cap_{k=m}^{\infty} F^0_{mk}$ and $E^0_G := \cap_{k=1}^{\infty} G^0_k$.

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[1] $\mathbb{N} = \{0,1,2,\dots\}$, and $\mathbb{N}^* = \{1,2,\dots\}$, the ISO 31-11 conventions.

[2] A colon-equals (:=) combination indicates a definition of the item on the left side.

1. The sets F_{mk} and G_m

For any set $S \subseteq \mathbb{R}$, we will use $\overline{S}$ and $|S|$ for the closure and Lebesgue measure of S, respectively. Under the conditions that Khinchin imposes, which we assume, we have sets with the following properties:

$$\left|F^0_{mk}\right| \to 0 \text{ as } k \to \infty, \quad \left|\overline{F^0_{mk}}\right| = \left|F^0_{mk}\right| \neq 0, \quad 0 \notin F^0_{mk}; \tag{1}$$

$$\left|G^0_k\right| \to 0 \text{ as } k \to \infty, \quad \left|\overline{G^0_k}\right| = \left|G^0_k\right| \neq 0, \quad 0 \notin G^0_k. \tag{2}$$

Discussion of these properties is deferred until the Section 5.

Extend F^0_{mk} and G^0_k to all intervals in $\mathbb{R}$ as follows. For positive integers m and k, let $\{n(j) \in \mathbb{N}^* \mid j \in \mathbb{Z}\}$ satisfy both

$$\left|F^0_{m,k+n(j)}\right| < 2^{-(|j|+k)}, \text{ and } \quad |G^0_{k+n(j)}| < 2^{-(|j|+k)},$$

which is possible because of the first property in each of (1) and (2). Translate the sets and combine them: let

$$F_{mk} := \bigcup_{j=-\infty}^{\infty} (j + F^0_{m,k+n(j)}), \quad \text{and} \quad G_k := \bigcup_{j=-\infty}^{\infty} (j + G^0_{k+n(j)}).$$

Since these are countable unions of sets in pairwise disjoint intervals, from the second property of each of (1) and (2), $\left|\overline{F_{mk}}\right| = |F_{mk}|$, and $\left|\overline{G_k}\right| = |G_k|$. Also

$$\left|F_{mk}\right| = \sum_{j=-\infty}^{\infty} \left|j + F^0_{m,k+n(j)}\right| < \sum_{j=-\infty}^{\infty} 2^{-(|j|+k)} = 3 \cdot 2^{-k}.$$

The same applies to G_k, so $|G_k| < 3 \cdot 2^{-k}$. Thus, we have the following extensions of (1) and (2):

$$\left|F_{mk}\right| \to 0 \text{ as } k \to \infty, \quad \left|\overline{F_{mk}}\right| = \left|F_{mk}\right|, \quad 0 \notin F_{mk}; \tag{3}$$

$$\left|G_k\right| \to 0 \text{ as } k \to \infty, \quad \left|\overline{G_k}\right| = \left|G_k\right|, \quad 0 \notin G_k. \tag{4}$$

Let

$$F_m := \bigcap_{k=m}^{\infty} F_{mk}, \quad F := \bigcup_{m=1}^{\infty} F_m, \text{ and } \quad G := \bigcap_{k=1}^{\infty} G_k.$$

Then $|F_m| = 0$, $|F| = 0$, and $|G| = 0$. Let $E := F \cup G$. Then $|E| = 0$. It is easy to establish that $E \supseteq \cup_{j=-\infty}^{\infty}(j + E^0_F \cup E^0_G)$, so $E \supseteq E_{\mathrm{K}}$.

We will construct nonnegative, bounded, continuous functions f_F and f_G which are zero except on a set of finite measure, and such that, for sufficiently large n, $f_F(n\,x) \geq 1/\sqrt{c_n}$ when $x \in F$, and $f_G(n\,x) \geq 1/\sqrt{c_n}$ when $x \in G$. Then $f_E := f_F + f_G$ will be integrable and $\lim_{n\to\infty} c_n f_E(n\,x) = \infty$ for $x \in E$. This is more than is necessary.

2. Auxiliary functions

Here we define several functions used in constructing f_F and f_G. The continuous function

$$u(x) := \begin{cases} 0 & \text{for } |x| < 1/2, \\ 2\,|x| - 1 & \text{for } 1/2 \leq |x| < 1, \\ 1 & \text{for } 1 \leq |x| \end{cases}$$

will be used to restrict the support of functions to be defined.

For each $m \in \mathbb{N}^*$, choose increasing sequences $\{k(m,l) \in \mathbb{N}^* \,|\, k(m,l) \geq m, l \in \mathbb{N}^*\}$ and $\{k(l) \in \mathbb{N}^* \,|\, l \in \mathbb{N}^*\}$, so that $k(m,l) \to \infty$ and $k(l) \to \infty$ as $l \to \infty$;

$$\left|F_{m,k(m,l)}\right| < \frac{1}{l}\, 2^{-(m+l+1)}; \text{ and } \quad \left|G_{k(l)}\right| < \frac{1}{l}\, 2^{-(l+1)}.$$

This is possible because of the first property in each of (3) and (4). Let $F^*_{m,k(m,l)}$ and $G^*_{k(l)}$ be an open sets such that

1. $\overline{F_{m,k(m,l)}} \subset F^*_{m,k(m,l)}$, and $\overline{G_{k(l)}} \subset G^*_{k(l)}$;
2. $|F^*_{m,k(m,l)}| < 2\,|F_{m,k(m,l)}|$, and $|G^*_{k(l)}| < 2\,|G_{k(l)}|$.

For example, because of the second property in each of (3) and (4), we can take $F^*_{m,k(m,l)}$ and $G^*_{k(l)}$ to be open sets determining the outer measure of $\overline{F_{m,k(m,l)}}$ and $\overline{G_{k(l)}}$, respectively. Note that

$$F_m \subseteq \bigcap_{l=1}^{\infty} F_{m,k(m,l)}, \text{ and } \quad G \subseteq \bigcap_{l=1}^{\infty} G_{k(l)}.$$

Let v_{Fml} and v_{Gl} be continuous functions with values in $[0,1]$, and

$$v_{Fml}(x) := \begin{cases} 0 \text{ for } x \notin F^*_{m,k(m,l)} \\ 1 \text{ for } x \in \overline{F_{m,k(m,l)}}, \end{cases}$$

and

$$v_{Gl}(x) := \begin{cases} 0 \text{ for } x \notin G^*_{k(l)} \\ 1 \text{ for } x \in \overline{G_{k(l)}}. \end{cases}$$

Urysohn's Lemma provides such functions. We see that $v_{Fml}(x) = 1$ for $x \in F_m$, and $v_{Gl}(x) = 1$ for $x \in G$.

3. The functions f_{Fm} and f_{Gm}

Henceforth assume, without loss of generality, that $\{c_n\}$ is positive and nondecreasing (otherwise, replace any $c_n \leq 0$ with 1, a finite number of replacements since $c_n \to \infty$, then replace every c_n with $\inf_{k \geq n} c_k$).

For $x \in \mathbb{R}$, and $m \in \mathbb{N}^*$, let

$$f_{Fm}(x) := \sup_{l \in \mathbb{N}^*} \left(\frac{1}{\sqrt{c_l}}\, v_{Fml}\Big(\frac{x}{l}\Big)\, u\Big(l^{-1} m\, x^2\Big)\, u\Big(\frac{x}{m}\Big) \right),$$

and

$$f_{Gm}(x) := \sup_{l \in \mathbb{N}^*} \left(\frac{1}{\sqrt{c_l}}\, v_{Gl}\Big(\frac{x}{l}\Big)\, u\Big(l^{-1} m\, x^2\Big)\, u\Big(\frac{x}{m}\Big) \right),$$

These functions have the following properties:

- P1. $0 \leq f_{Fm}(x) \leq 1/\sqrt{c_1}$, and $0 \leq f_{Gm}(x) \leq 1/\sqrt{c_1}$.
- P2. f_{Fm} and f_{Gm} are continuous.
- P3. If $n \in \mathbb{N}^*$, $m\, n\, x^2 \geq 1$, and $n\,|x| \geq m$; then $f_{Fm}(n\,x) \geq 1/\sqrt{c_n}$ if $x \in F_m$, and $f_{Gm}(n\,x) \geq 1/\sqrt{c_n}$ if $x \in G$.
- P4. For x in any bounded set, $f_{Fm}(x) = 0$ and $f_{Gm}(x) = 0$ except for a finite number of values of m.
- P5. $\{x \,|\, f_{Fm}(x) \neq 0\} \subseteq \bigcup_{l=1}^{\infty} (l\, F^*_{m,k(m,l)})$, and $\{x \,|\, f_{Gm}(x) \neq 0\} \subseteq \bigcup_{l=1}^{\infty} (l\, G^*_{k(l)})$.
- P6. $\big|\{x \,|\, f_{Fm}(x) \neq 0\}\big| \leq 2^{-m}$, and $\big|\{x \,|\, f_{Gm}(x) \neq 0\}\big| \leq 2^{-m}$.

The justifications for these are the following:

1. $1/\sqrt{c_l} \le 1/\sqrt{c_1}$ since c_n is nondecreasing.
2. Locally each of f_{Fm} and f_{Gm} is the maximum of a finite number of continuous functions: if $|x| < M$, then $u(l^{-1}m\,x^2) = 0$ if $l > 2\,m\,M^2$.
3. When $l = n$, $n\,x/l = x$, so the following hold: $l^{-1}m\,(n\,x)^2 = m\,n\,x^2 \ge 1$ so $u\big(l^{-1}m\,(n\,x)^2\big) = 1$; $u(n\,x/m) = 1$; if $x \in F_m$, $v_{Fml}(n\,x/l) = 1$, so $f_{Fm}(n\,x) \ge 1/\sqrt{c_l} = 1/\sqrt{c_n}$; and if $x \in G$, $v_{Gl}(n\,x/l) = 1$, so $f_{Gm}(n\,x) \ge 1/\sqrt{c_l} = 1/\sqrt{c_n}$.
4. For $|x| < M$, $u(x/m) = 0$ for $m \ge 2\,M$.
5. $\{x \mid v_{Fml}(x/l) \ne 0\} \subseteq l\,F^*_{m,k(m,l)}$, and $\{x \mid v_{Gm}(x/l) \ne 0\} \subseteq l\,G^*_{k(l)}$.
6. $\big|\{x \mid f_{Fm}(x) \ne 0\}\big| \le \sum_{l=1}^{\infty} l\,\big|F^*_{m,k(m,l)}\big| < \sum_{l=1}^{\infty} l \cdot 2\,|F_{m,k(m,l)}|$
$$< \textstyle\sum_{l=1}^{\infty} l \cdot 2 \cdot l^{-1}\,2^{-(m+l+1)} = 2^{-m},$$
and similarly for f_{Gm}.

4. Proof of the theorem

Let
$$f_F(x) := \sup_{m\in\mathbb{N}^*} f_{Fm}(x), \text{ and } \quad f_G(x) := \sup_{m\in\mathbb{N}^*} f_{Gm}(x).$$
By P1, P2, and P4, f_F is nonnegative, continuous and bounded. Moreover
$$\{x \mid f_F(x) \ne 0\} = \bigcup_{m=1}^{\infty} \{x \mid f_{Fm}(x) \ne 0\}$$
so, by P6,
$$\big|\{x \mid f_F(x) \ne 0\}\big| < \sum_{m=1}^{\infty} 2^{-m} = 1.$$
Therefore, f_F is bounded, and it is zero except on a set of finite measure, hence, it is integrable on $\mathbb{R}$. That $f_F(n\,x) \ge 1/\sqrt{c_n}$ for $x \in F$ and large enough n follows from property P3 since x will be in some F_m. Thus
$$\lim_{n\to\infty} c_n\, f_F(n\,x) \ge \lim_{n\to\infty} \sqrt{c_n} = \infty.$$
The same applies, *mutatis mutandis*, to f_G.

Thus, $f = f_E + f_{E'} = f_F + f_G + f_{E'}$ satisfies the requirements and this completes the proof of the theorem. $\square$

5. Properties of F^0_{mk} and G^0_k

Khinchin establishes the first property in both (1) and (2). Proof of other properties in (1) and (2) requires understanding the definitions of the sets F^0_{mk} and G^0_m, for which we need a few facts about continued fractions.

The notation $\alpha(n) = [a_0; a_1, a_2, \ldots, a_n]$ will be used for the *terminating continued fraction*
$$\alpha(n) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n}}}}$$
with every numerator equal to 1. The numbers $a_0, \ldots, a_n$ are called the *elements* of $\alpha(n)$. Henceforth, $a_0 \in \mathbb{Z}$; $a_k \in \mathbb{N}^*$ for $1 \le k < n$; and $a_n > 1$. The latter is done

for uniqueness without loss of generality: if $a_n = 1$, remove it and replace a_{n-1} with $a_{n-1}+1$. A *nonterminating continued fraction* α has the same form except that the term with a_n is absent. It will be convenient to extend the notation as follows: for a terminating $\alpha(n)$, let $a_i = 0$ for $i > n$, and use the nonterminating form

$$\alpha(n) = [a_0; a_1, \ldots, a_n, a_{n+1}, \ldots] = [a_0; a_1, \ldots, a_n, 0, 0, \ldots].$$

In particular, if $a_i = 0$ for $i \geq 1$, $[a_0; 0, 0, \ldots] = a_0$.

A terminating continued fraction is a simple fraction:

$$\alpha(n) = \frac{p_n}{q_n}; \tag{5}$$

p_i and q_i are integers defined by

$$\begin{aligned} p_{-1} &:= 1, \quad p_0 := a_0, \quad q_{-1} := 0, \quad q_0 := 1, \\ &\left.\begin{aligned} p_i &:= a_i\, p_{i-1} + p_{i-2} \\ q_i &:= a_i\, q_{i-1} + q_{i-2} \end{aligned}\right\} \quad \text{for } i = 1, \ldots \end{aligned} \tag{6}$$

[2, Theorem 1]. For $i \geq 1$ they are increasing functions of i when $a_i \neq 0$, and

$$\frac{p_i}{q_i} = [a_0; a_1, \ldots, a_i]$$

is the *convergent of rank* i of $\alpha(n)$. We identify any $\alpha \in \mathbb{R}$ with its unique continued fraction; if α is irrational, the continued fraction is nonterminating and its convergents tend to α.

For any α, terminating or not, the representation can be truncated: for $i \geq 1$, $\alpha = \alpha(i, r_i)$, where

$$\alpha(i, r_i) := [a_0; a_1, \ldots, a_i, r_i], \tag{7}$$

with $r_i := a_{i+1} + [0; a_{i+2}, \ldots]$. If $r_{i+1} \neq 0$ (i.e., $a_{i+2} \neq 0$), $r_i = a_{i+1} + 1/r_{i+1}$, so $a_{i+1} < r_i < a_{i+1}+1$. If α is nonterminating, r_i can be any number in $(1, \infty)$. From (5) and (6) we have

$$\alpha(i, r_i) = \frac{r_i\, p_i + p_{i-1}}{r_i\, q_i + q_{i-1}}. \tag{8}$$

In (6), multiply p_i by q_{i-1} and q_i by p_{i-1}, then subtract the results to obtain a recursion which yields

$$p_{i-1}\, q_i - p_i\, q_{i-1} = (-1)^i \quad \text{for } i \geq 0 \tag{9}$$

[2, Theorem 2].

We will limit our attention to the interval $(0, 1)$; that is, take $a_0 = 0$. According to (8), the range of $\alpha(i, r_i)$ for $r_i \in (1, \infty)$ is an open interval $J_{\alpha(i)}$ with endpoints

$$\frac{p_i}{q_i}, \quad \text{and} \quad \frac{p_i + p_{i-1}}{q_i + q_{i-1}}.$$

This interval is all numbers of the form $[0; a_1, \ldots, a_i, a_{i+1}, \ldots]$, with specified $\alpha(i) = [0; a_1, \ldots, a_i]$, and any $a_{i+1} \in \mathbb{N}^*$, whether terminating or not. Thus $0 \notin J_{\alpha(i)}$.

Let $\{\phi(i) \in \mathbb{R} \mid i \in \mathbb{N}^* \text{ and } \phi(i) > 1\}$ be a nondecreasing sequence, and use it to constrain the elements of $\alpha(m+k, r_{m+k})$ by the condition

$$a_{m+i} < \phi(m+i) \text{ for } i = 1, \ldots, k; \tag{10}$$

there are no constraints on $a_1, \dots, a_m$ (see Theorem 30 in Khinchin's book). The set of all numbers in (0,1) satisfying condition (10) and $1 < r_{m+k} < \infty$ is the open set

$$F^0_{mk} := \bigcup_{\alpha(m+k)} J_{\alpha(m+k)}, \tag{11}$$

where the union is over all $\alpha(m+k)$ satisfying (10). Thus $0 \notin F^0_{mk}$, and $|F^0_{mk}| \neq 0$.

For $k \in \mathbb{N}^*$ and A a positive number satisfying $A - \ln A - \ln 2 - 1 > 0$, the set of numbers $\alpha(k, r_k) \in (0,1)$ satisfying the product bound

$$a_1 \cdots a_k \geq e^{A\,k}, \tag{12}$$

and $1 < r_k < \infty$ is

$$G^0_k := \bigcup_{\alpha(k)} J_{\alpha(k)},$$

where the union is over all $\alpha(k)$ satisfying (12) (see the proof of Theorem 31 in Khinchin's book). Thus $0 \notin G^0_k$, and $|G^0_k| \neq 0$.

It is not true that $\left|\overline{S}\right| = |S|$ for any set $S \subset \mathbb{R}$, even if S is a countable union of pairwise disjoint open intervals. The construction of a Cantor set with nonzero measure provides a counterexample.

Here we have special sets, $S = F^0_{mk}$ or $S = G^0_k$, which are countable unions of pairwise disjoint open sets. The closure $\overline{S}$ is the union of the closures of the open sets and a set L. The latter comprises the limits of sequences in S but not ultimately in a single one of the open sets. Now we will establish that L is countable, and that $|\overline{S}| = |S|$. The continued-fraction construction provides an ordering of the open intervals that makes this possible.

For this, we consider a sequence $\alpha_n = [0; a_{n1}, a_{n2}, \dots] \in S$, and let that $\alpha_n \to \alpha = [0; a_1, a_2, \dots]$ as $n \to \infty$. We will use the truncated notations

$$\alpha_n = [0; a_{n1}, \dots, a_{ni}, r_{ni}] \quad \text{and} \quad \alpha = [0; a_1, \dots, a_i, r_i].$$

It is apparent that, for small values of j, the condition $\alpha_n \to \alpha$ forces the elements a_{nj} to become independent of n as n becomes large. The following lemma is a precise statement of that.

Lemma 1. *If, for some $i \in \mathbb{N}^*$, $a_{n1}, \dots, a_{ni}$ are bounded, and there are numbers $R_{i,min}$ and $R_{i,max}$ such that $1 < R_{i,min} \leq r_{ni} \leq R_{i,max} < \infty$, then there is a number $n_0 \in \mathbb{N}^*$ such that, for $1 \leq j \leq i$, a_{nj} is independent of n for $n > n_0$, and $a_j = \lim_{\nu\to\infty} a_{\nu j}$; for sufficiently large n, $a_j = a_{nj}$.*

Proof. The lower bound on r_{ni} guarantees that α_n does not terminate before $a_{n,i+2}$, and $1 \leq a_{n,i+1} \leq r_{ni} \leq a_{n,i+1} + 1$.

If, for a moment, we regard $a_1, \dots, a_i$, and r_i as real variables rather than integers, we can differentiate α_n with respect to a_{ni}, using (6), (8), and (9):

$$\left|\frac{\partial\,\alpha_n}{\partial\,a_{ni}}\right| = \left|\frac{r^2_{ni}\,(p_{n,i-1}\,q_{ni} - p_{ni}\,q_{n,i-1})}{(r_{ni}\,q_{ni} + q_{n,i-1})^2}\right| = \left|\frac{r^2_{ni}\,(-1)^i}{(r_{ni}\,q_{ni} + q_{n,i-1})^2}\right| \geq \epsilon_i > 0,$$

where $\epsilon_i = (q_{ni} + q_{n,i-1}/R_{i,min})^{-2}$.

Therefore, again considering the variables to be integers, if $a_{ni} \neq a_{n+1,i}$, then $|a_{ni} - a_{n+1,i}| \geq 1$, so $|\alpha_n - \alpha_{n+1}| \geq \epsilon_i$. This cannot happen for large n since $\alpha_n \to \alpha$, so for sufficiently large n, a_{ni} must be independent of n.

This can be applied to smaller values of i, so all of $a_{n1}, \ldots, a_{ni}$ are independent of n for sufficiently large n; i.e., for $n > n_0$, this determining n_0.

Since α_n does not terminate at a_{ni}, $\alpha_n \neq p_{ni}/q_{ni}$, and we can solve (8) for r_{ni}:

$$r_{ni} = \frac{-\alpha_n\, q_{n,i-1} + p_{n,i-1}}{\alpha_n\, q_{ni} - p_{ni}}.$$

Since r_{ni} is bounded, this is a continuous function of α_n, so r_{ni} approaches a limit r_i^*, and $\alpha = \lim_{\nu\to\infty} \alpha_\nu = [a_{n1}, \ldots, a_{ni}, r_i^*]$ for $n > n_0$. As above, $|a_{nj} - a_j|$ must be less than 1 for large n, so $a_j = \lim_{\nu\to\infty} a_{\nu j} = a_{nj}$ for $1 \le j \le i$ if $n > n_0$. ☐

Lemma 2. *For some $i \in \mathbb{N}^*$, let $a_{n1}, \ldots, a_{ni}$ be bounded, and let $R_{i,min}$ and $R_{i,max}$ be such that $1 < R_{i,min} \le r_{ni} \le R_{i,max} < \infty$. If there is a number $n_0 \in \mathbb{N}^*$, and a function h with $0 \le h(a_{n1}, \ldots, a_{ni})$ for all $n > n_0$, then $0 \le h(a_1, \ldots, a_i)$.*

Also, for $1 \le j \le i$, if there is a number M_j such that, $a_{nj} < M_j$ for $n \ge n_0$, then $a_j < M_j$.

Proof. This follows from Lemma 1 because $a_j = a_{nj}$ for sufficiently large n. ☐

If $\{r_{ni}\}$ is not bounded, we cannot conclude that a_{ni} is independent of n. Indeed, it might be that one of $r_{2\,n,i}$ and $r_{2\,n+1,i}$ tends to 1, the other to ∞, in which case $a_{2\,n+1,i} - a_{2\,n,i}$ tends to 1 or -1. For example, $\alpha_{2n} = [0; 2, n]$ and $\alpha_{2n+1} = [0; 1, 1, n]$, both of which tend to $[0; 2] = 1/2$, and for which $r_{2n,1} = n \to \infty$, and $r_{2n+1,1} = 1 + 1/n \to 1$.

Lemma 3. *For some $i \in \mathbb{N}^*$, let $a_{n1}, \ldots, a_{ni}$ be bounded. If r_{ni} is not bounded away from 1 or ∞, then then α terminates; i.e, $\alpha = [0; a_1, \ldots, a_j]$ for some $j \le i$. If $a_1 \to \infty$, $\alpha = 0$ (corresponding to $i = j = 0$).*

Proof. From Lemma 1, $a_{n1}, \ldots, a_{n,i-1}$ are constant for sufficiently large n. For those values of n, $p_{i-2} := p_{n,i-2}$, $p_{i-1} := p_{n,i-1}$, $q_{i-2} := q_{n,i-2}$, and $q_{i-1} := q_{n,i-1}$ are independent of n, so

$$\alpha_n = \frac{r_{n,i-1}\, p_{i-1} + p_{i-2}}{r_{n,i-1}\, q_{i-1} + q_{i-2}}.$$

Suppose r_{ni} is not bounded away from ∞. Since a_{ni} is a bounded integer, and therefore has only a finite number of limit points, there is a subsequence of α_n with $r_{ni} \to \infty$, and a_{ni} converging to one of those limit points; call the limit point a_i^*. For that subsequence, $r_{n,i-1} = a_{ni} + 1/r_{n,i} \to a_i^*$, so

$$\alpha = \frac{a_i^*\, p_{i-1} + p_{i-2}}{a_i^*\, q_{i-1} + q_{i-2}},$$

Thus, $\alpha = [0; a_1, \ldots, a_{i-1}, a_i^*]$. If $a_i^* = 1$, by our convention we change this to $\alpha = [0; a_1, \ldots, a_{i-2}, a_{i-1} + 1]$.

If r_{ni} is not bounded away from 1, the same applies except that we use a subsequence with $r_{ni} \to 1$, so $r_{n,i-1} = a_{ni} + 1/r_{ni} \to a_{ni} + 1$. In this case $\alpha = [0; a_1, \ldots, a_{i-1}, a_i + 1]$.

That $\alpha = 0$ if $a_1 \to \infty$ is obvious. ☐

Lemma 4. $\left|\overline{F_{mk}^0}\right| = |F_{mk}^0|$, *and* $\left|\overline{G_k^0}\right| = |G_k^0|$.

Proof. Equation (11) shows that the set F_{mk}^0 is the union of a countable set of pairwise disjoint open intervals $J_{\alpha(m+k)}$. We will show that there is a countable set

L_{mk} such that $\overline{F^0_{mk}} = L_{mk} \bigcup \Big(\bigcup_{\alpha(m+k)} \overline{J_{\alpha(m+k)}} \Big)$. Since $\left|\overline{J_{\alpha(m+k)}}\right| = |J_{\alpha(m+k)}|$, it follows that $\left|\overline{F^0_{mk}}\right| = |L_{mk}| + \sum_{\alpha(m+k)} \left|\overline{J_{\alpha(m+k)}}\right| = 0 + \sum_{\alpha(m+k)} |J_{\alpha(m+k)}| = |F^0_{mk}|$.

Apply the results above with $S = F^0_{mk}$ and $M_j = \phi(j)$ for $m+1 \le j \le m+k$. If $a_{n1}, \ldots, a_{nm}$ are bounded, and $r_{n,m+k}$ is bounded away from 1 and ∞, then α satisfies (10) by Lemma 2, so it is in some $J_{\alpha(m+k)}$, hence in $\overline{J_{\alpha(m+k)}}$. Limits of the remaining sequences constitute the set L_{mk}. By Lemma 3, each point in this set is zero or it has the form $[0; a_1, \ldots, a_j]$, where $1 \le j \le m+k$. Therefore, L_{mk}, is countable because the set of combinations $\{a_1, \ldots, a_{m+k}\}$ is countable.

For $S = G^0_k$, let $h(a_1, \ldots, a_k) = a_1 \cdots a_k - e^{A\,k}$. If $a_{n1}, \ldots, a_{nk}$ are bounded, and r_{mk} is bounded away from 1 and ∞, apply the Lemma 2 to see that α satisfies (12), so α is in some $J_{\alpha(k)}$, hence in $\overline{J_{\alpha(k)}}$. As above, limits of the remaining sequences constitute a set L_k which is countable, and it follows that $\left|\overline{G^0_k}\right| = |G^0_k|$. $\square$

Thus we have established the second and third properties in both (1) and (2).

6. Appendix

For convenient reference we provide

Khinchin's Theorem. [2, Theorem 32] *Let $\{b_n > 0 \,|\, n \in \mathbb{N}^*\}$ have $n\, b_n$ nonincreasing for sufficiently large n, and $\sum_n b_n$ divergent. Then for almost all $\alpha \in \mathbb{R}$ there are infinitely-many pairs $\{m, n\} \subset \mathbb{N}^*$ such that*

$$|n\,\alpha - m| < b_n.^3 \tag{13}$$

Remark 1. It is not clear how the elements of $\alpha \in (j, j+1)$ change when α is negated. For $j \ge 0$, the negative of $[j; a_1, a_2, \ldots]$ is $-j - [0; a_1, a_2, \ldots] = [-j-1; a'_1, a'_2, \ldots]$. The elements must satisfy $a'_k \ge 1$, and it might be that conditions such as $a_i \le \phi_i$ are not preserved under such a change. For example, a_1 increases in the terminating continued fractions

$$\frac{3}{4} = [0; 1, 3] \quad \text{and} \quad -\frac{3}{4} = -1 + \frac{1}{4} = [-1; 4].$$

Thus it seems likely that E is not symmetric (i.e., that $-E \ne E$).

Remark 2. Consider E_{K}, with the subscript chosen in honor of Khinchin, as the smallest possible zero-measure set of Khinchin's Theorem. It must be symmetric. Therefore it is likely that $E_{\mathrm{K}} \ne E$. Note that E_{K} is not a unique set, but that it depends on $\{b_n\}$.

References

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[2] A. Ya. Khinchin, *Continued Fractions*, (translated by Peter Wynn), P. Noordhoff, Groningen, The Netherlands, 1963. Also available under the same title (but with a different translation from the Russian) by Dover Publications, Inc., Mineola, NY, 1997, (copyright 1964 and 1992 by the University of Chicago), ISBN-10: 0-486-69630-8, ISB-13: 978-0-486-69630-0.

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[3] Khinchin uses the Borel-Cantelli Lemma to show also that if $\sum_n b_n$ converges, then (13) is satisfied for only a finite number of pairs $\{m, n\}$, except for α in a set of measure zero.